Distance Preservation for All Polynomial Generators

Sarah Bordage and Alessandro Chiesa
EPFL

Lattices Meet Hashes
May 2, 2023

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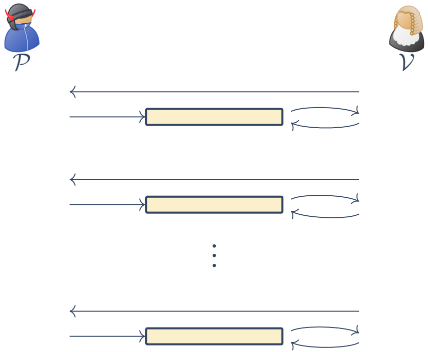
$$\min_{c \in \mathscr{C}} \Delta(u, c) = \Delta(u, \mathscr{C}) < \delta.$$

Otherwise, u is **δ-far from $\mathscr{C}$**.

Probabilistic proofs and proximity testing to codes

Interactive Oracle Proofs + **Cryptographic Hash Functions** → **Post-Quantum ZK-SNARKs**



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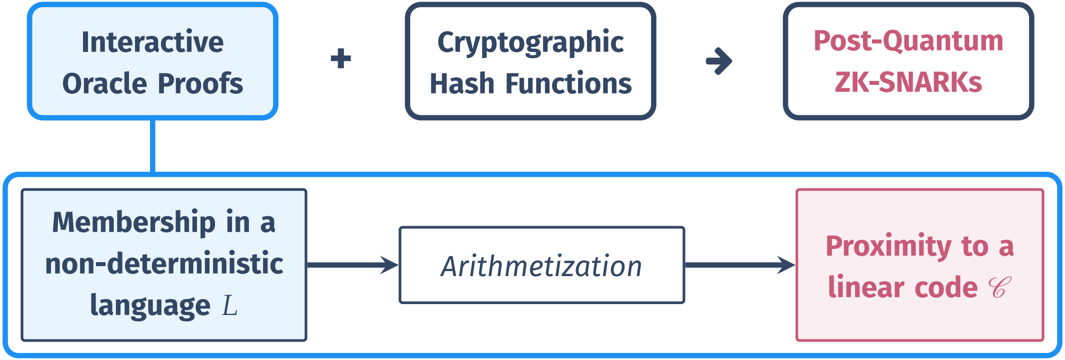
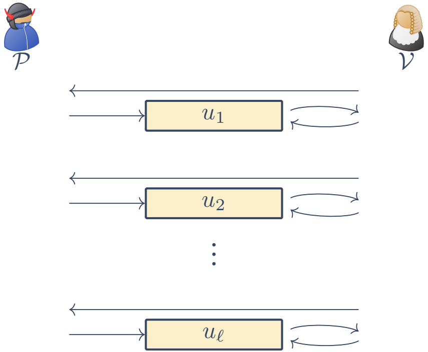


Batch Proximity Testing in Interactive Oracle Proofs

- If x $\in L$, then $\exists u_1, \ldots, u_\ell \in \mathscr{C}$ satisfying all verifier's checks.
- If x $\notin L$, then any $(u_1, \ldots, u_\ell) \in (\mathbb{F}^n)^\ell$ falsifies verifier's checks with high probability, **given that the u_i's are all close to $\mathscr{C}$**.



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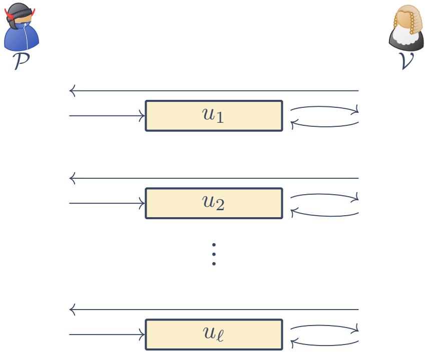

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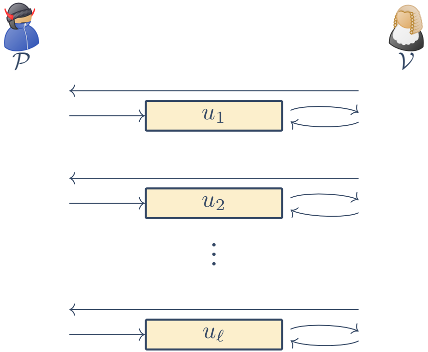

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Proximity tests can be **expensive**, e.g. FRI protocol used in STARKs, Aurora, Ligero, Shockwave, ...

Testing Proximity to Linear Codes

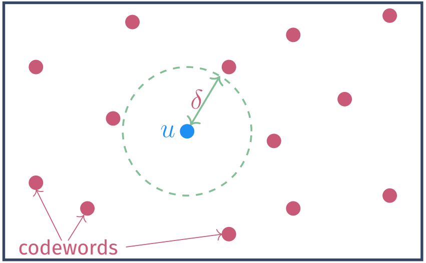
Proximity test for a single vector

Proximity test $(\mathcal{P}, \mathcal{V})$

Given:
- linear code $\mathscr{C} \subseteq \mathbb{F}^n$
- proximity parameter δ
- purported codeword $u \in \mathbb{F}^n$

$\mathcal{P}$'s inputs: $\mathscr{C}, \delta, u$.

$\mathcal{V}$'s inputs: $\mathscr{C}, \delta$ and oracle access to u.



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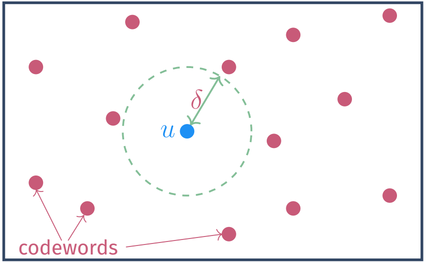

Completeness. If $u \in \mathscr{C}$, verifier $\mathcal{V}$ accepts.

Soundness. If $\Delta(u, \mathscr{C}) \geq \delta$, verifier $\mathcal{V}$ rejects with high prob.

Batch proximity test using random linear combinations [RVW13]

Batch proximity test $(\mathcal{P}_{\text{batch}}, \mathcal{V}_{\text{batch}})$

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Given:
- linear code $\mathscr{C} \subseteq \mathbb{F}^n$
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1. $\mathcal{V}_{\text{batch}} \to \mathcal{P}_{\text{batch}} : (z_1, \ldots, z_\ell) \xleftarrow{\$} \mathbb{F}^\ell$.
2. $\mathcal{P}_{\text{batch}}$ and $\mathcal{V}_{\text{batch}}$ run $(\mathcal{P}, \mathcal{V})$ to check δ-proximity of $\sum z_i u_i$ to $\mathscr{C}$.

Key properties:

- If $u_1, \ldots, u_\ell \in \mathscr{C}$, then $\sum z_i u_i \in \mathscr{C}$.
- For every $\delta \in (0, \frac{1}{2})$, if $\max_i \Delta(u_i, \mathscr{C}) \geq \delta$, then

$$\Pr_{z_1,\ldots,z_\ell \leftarrow \mathbb{F}^\ell} \left[\Delta\left(\sum z_i u_i, \mathscr{C}\right) < 2\delta\right] \leq \frac{1}{|\mathbb{F}|}.$$

Correlated agreements

Many situations require a **stronger guarantee**.

If there are many $(z_1, \dots, z_\ell) \in \mathbb{F}^\ell$ such that $\sum z_i u_i$ is close to $\mathscr{C}$, it must be because $u_1, \dots, u_\ell \in \mathbb{F}^n$ have large **correlated agreement** with the code $\mathscr{C}$:

$$\exists T \subseteq [n], \exists c_1, \dots, c_\ell \in \mathscr{C} \text{ s.t. } \begin{cases} |T| > (1-\delta)n, \\ \forall i \in [\ell], u_{i\,|T} = c_{i\,|T}. \end{cases}$$

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- **Example 1.** Soundness of IOP system requires oracles $u_1, \ldots, u_\ell$ to be close to different codes $\mathscr{C}_1, \ldots, \mathscr{C}_\ell$ with different rates.
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- **Example 2.** Soundness analysis of IOPs of Proximity for linear codes. [BBHR18, BKS18, BGKS20, BCIKS20, BCG20, ABN22, **B**LNR22]

Correlated agreement = proximity to interleaved code

Vectors $u_1, \ldots, u_\ell \in \mathbb{F}^n$ have large **correlated agreement** with $\mathscr{C}$:

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Interleaved code

$$\mathscr{C}^\ell := \left\{ \boldsymbol{C} = \begin{pmatrix} -c_1- \\ \vdots \\ -c_\ell- \end{pmatrix} \in \mathbb{F}^{\ell \times n} : \forall i \in [\ell], c_i \in \mathscr{C} \right\}$$

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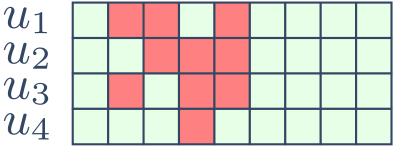
$$\text{Correlated agreement} \quad \longleftrightarrow \quad \Delta_{\mathbb{F}^\ell}(\boldsymbol{U}, \mathscr{C}^\ell) < \delta$$

Maximum distance vs column-wise distance

$$\mathscr{C} \subseteq \mathbb{F}^9 \qquad \boldsymbol{U} := \begin{pmatrix} -u_1- \\ -u_2- \\ -u_3- \\ -u_4- \end{pmatrix} \in \mathbb{F}^{4\times 9}$$

Green = correct

Red = error



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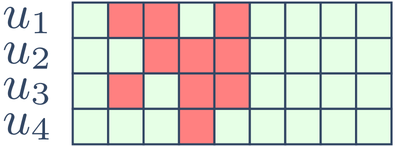

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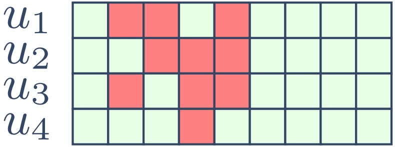

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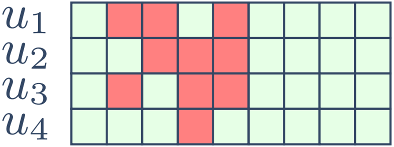

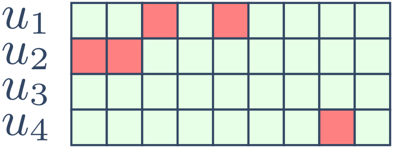

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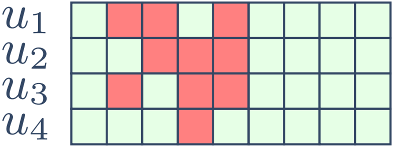

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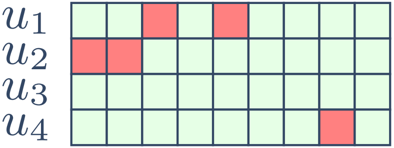

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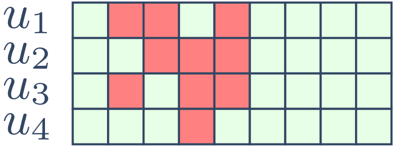

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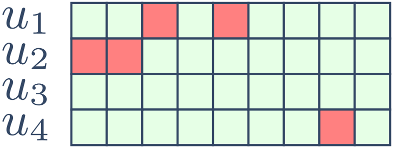

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Distance Preservation to Interleaved Codes

Distance preservation with random linear combinations

Distance preservation. There exists Λ s.t. for every $\delta \in (0, \Lambda)$,

$$\Delta_{\mathbb{F}^\ell}(\boldsymbol{U}, \mathscr{C}^\ell) \geq \delta \implies \Pr_{\boldsymbol{z} \leftarrow \mathbb{F}^\ell}\left[\Delta\left(\boldsymbol{z} \cdot \boldsymbol{U}, \mathscr{C}\right) < \sigma(\delta)\right] \leq \tau.$$

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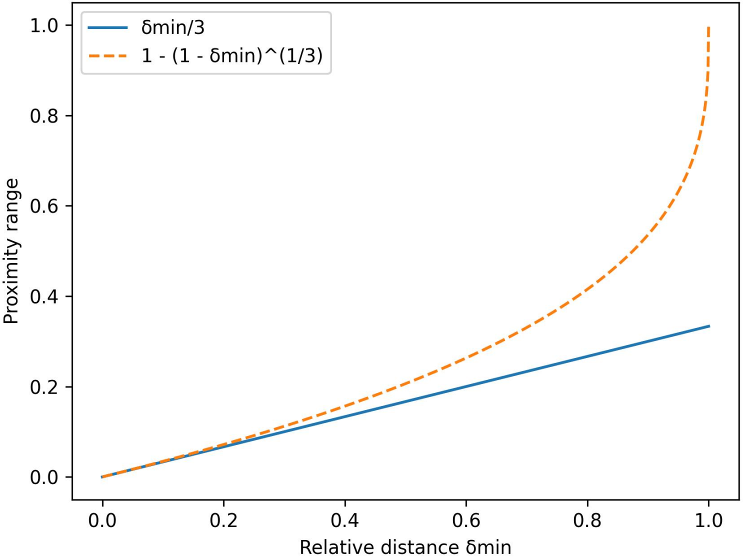
δmin/3
1 - (1 - δmin)^(1/3)
Proximity range
Relative distance δmin
0.0
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0.4
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- Better parameters for **specific** family of codes (Reed-Solomon) [BCIKS20].

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Why reduce randomness complexity?

- concrete efficiency of IOPs used in real-world (e.g. FRI, STARKs)
- sometimes necessary, e.g. IOPs with linear-time prover [BCL22, BCGL22]

(Recall) Goal : Batch proximity testing

We are looking for generators $G\colon \mathbb{F}^s \to \mathbb{F}^\ell$ that allow randomness-efficient batch proximity testing.

(Recall) Goal : Batch proximity testing

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Batch proximity test $(\mathcal{P}_{\text{batch}}, \mathcal{V}_{\text{batch}})$

Given:
- linear code $\mathscr{C} \subseteq \mathbb{F}^n$
- proximity parameter δ
- purported codewords $u_1, \ldots, u_\ell \in \mathbb{F}^n$ (oracles)

1. $\mathcal{V}_{\text{batch}} \to \mathcal{P}_{\text{batch}} : \boldsymbol{x} \xleftarrow{\$} \mathbb{F}^s$.
2. $\mathcal{P}_{\text{batch}}$ and $\mathcal{V}_{\text{batch}}$ run $(\mathcal{P}, \mathcal{V})$ to check δ-proximity of $\sum G(\boldsymbol{x})_i u_i$ to $\mathscr{C}$.

Distance-Preserving Generators

Warm-up: Epsilon-biased generators

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Parameters: $\Lambda \in (0,1)$, $\sigma\colon (0,1) \to (0,1)$ non-increasing fct, $\tau \in (0,1)$.

A function $G\colon \mathbb{F}^s \to \mathbb{F}^\ell$ is a (Λ,σ,τ)-**distance-preserving generator** if for every code $\mathscr{C} \subseteq \mathbb{F}^n$ and every $\delta \in (0,\Lambda)$:

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$\mathbb{F}$	$G(x) = (1, x, \ldots, x^{\ell-1})$	$\frac{\ell-1}{\lvert\mathbb{F}\rvert}$	✔ [BKS18]
$\mathbb{F}^s, 2^s = \ell$	$G(\boldsymbol{x}) = (\prod_i x_i^{b_i})_{b \in \{0,1\}^s}$	$\frac{s}{\lvert\mathbb{F}\rvert}$	✔ [ABN22]

From prior work: known distance-preserving generators are in particular biased.

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Easy fact: G is (Λ,σ,τ)-distance-preserving $\implies$ G is τ-biased.
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Question: Do all biased generators preserve distance?

Polynomial Generators Preserve Distance

Polynomial generators

Let s, ℓ, d be positive integers such that $d \leq |\mathbb{F}|$ and $\max(s, 2) \leq \ell \leq \binom{s+d}{s}$.

Polynomial generator

A function $G \colon \mathbb{F}^s \to \mathbb{F}^\ell$ is a *degree-d generator* if there exist ℓ linearly independent polynomials $P_1, \ldots, P_\ell \in \mathbb{F}[X_1, \ldots, X_s]$ of total degree at most d such that

$$\forall \boldsymbol{x} \in \mathbb{F}^s, \qquad G(\boldsymbol{x}) = (P_i(\boldsymbol{x}))_{1 \leq i \leq \ell}.$$

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- Any degree-d generator is ε–biased with $\varepsilon = \frac{d}{|\mathbb{F}|}$. (Schwartz-Zippel)

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- Any degree-d generator is ε–biased with $\varepsilon = \frac{d}{|\mathbb{F}|}$. (Schwartz-Zippel)
- Distance-preserving generators from literature are special cases of polynomial generators.

Main result

Theorem

Any degree-d generator $G\colon \mathbb{F}^s \to \mathbb{F}^\ell$ is (Λ, σ, τ)–distance-preserving.

	Proximity range Λ	**New distance** $\sigma(\delta)$	**Error** τ
Unique-decoding	$\frac{\delta_{\min}}{d+2}$	δ	$\delta n \cdot \frac{d}{\lvert\mathbb{F}\rvert}$
List-decoding	$1 - \sqrt[d+2]{1 - \delta_{\min} + \eta}$	δ	$\delta n \cdot \frac{\ell+1}{\eta} \cdot \frac{d}{\lvert\mathbb{F}\rvert}$

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- **Implies prior results** about distance-preserving generators
- **Improves prior results**
  - For $G(x) = (x^i)_{0 \le i < \ell}$, **remove** from τ the **exponential dependence** in ℓ from [BKS18]
  - **Exact** distance preservation (instead of *approximate*)

Application: Proximity gaps for all linear codes

Theorem $\implies$ Proximity gaps for all linear codes

Let $\delta \in (0, \Lambda)$. Let $\mathscr{C}$ be a linear code and let $G \colon \mathbb{F}^s \to \mathbb{F}^\ell$ be a polynomial generator. Exactly one of the following two statements holds:

$$(1)\, \Pr\left[\Delta\left(G(\boldsymbol{x})^\top \cdot \boldsymbol{U}, \mathscr{C}\right) < \delta\right] = 1 \quad \textbf{OR} \quad (2)\, \Pr\left[\Delta\left(G(\boldsymbol{x})^\top \cdot \boldsymbol{U}, \mathscr{C}\right) < \delta\right] \leq \tau.$$

Previous work on proximity gaps:

- All linear codes – uniform coefficients, $\delta < \frac{\delta_{\min}}{3}$ [AHIV17, RZ17]
- RS codes – uniform coefficients & powers, $\delta < 1 - \sqrt{1 - \delta_{\min}}$ [BCIKS20]

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In fact, nearly all combinations are at the same distance.

If $\Delta_{\mathbb{F}^\ell}(\boldsymbol{U}, \mathscr{C}^\ell) \in (0, \Lambda)$, then $\Pr\left[\Delta(G(\boldsymbol{x})^\top \cdot \boldsymbol{U}, \mathscr{C}) \neq \Delta_{\mathbb{F}^\ell}(\boldsymbol{U}, \mathscr{C}^\ell)\right] \leq \tau$.

Technical Overview

Proof overview

Theorem

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Any **univariate** degree-d generator $G \colon \mathbb{F} \to \mathbb{F}^\ell$ is (Λ, σ, τ)-distance-preserving.

⬇

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Proof overview

Generators from MDS codes are distance-preserving.

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Epsilon-biased generators from codes with good distance

Generators from linear codes

Let $\mathcal{L} \subseteq \{\mathbb{F}^s \to \mathbb{F}\}$ be a $\mathbb{F}$-linear space and let $\{f_1, \ldots, f_\ell\}$ be a basis of $\mathcal{L}$.

Epsilon-biased generators from codes with good distance

Generators from linear codes

Let $\mathcal{L} \subseteq \{\mathbb{F}^s \to \mathbb{F}\}$ be a $\mathbb{F}$-linear space and let $\{f_1, \ldots, f_\ell\}$ be a basis of $\mathcal{L}$.

Consider the evaluation map $\mathrm{ev}\colon \begin{array}{ll} \mathcal{L} & \to \mathbb{F}^N \\ f & \mapsto (f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{F}^s) \end{array}$, where $N := |\mathbb{F}|^s$.

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We have:

- $\mathscr{D} = \mathrm{ev}(\mathcal{L})$ is a $[N, \ell]$-code.

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- $\mathscr{D} = \mathrm{ev}(\mathcal{L})$ is a $[N, \ell]$-code.
- If $\delta_{\min}(\mathscr{D}) \geq 1 - \varepsilon$, then $G_{\mathscr{D}}\colon \begin{array}{ll} \mathbb{F}^s & \to \mathbb{F}^\ell \\ \boldsymbol{x} & \mapsto (f_i(\boldsymbol{x}))_{i \in [\ell]} \end{array}$ is ε-biased.

A small-biased generator from a Reed-Solomon code

Example

Let $\ell \leq |\mathbb{F}|$. Consider the encoding map $\mathrm{ev} : \begin{array}{ll} \mathbb{F}[x]_{<\ell} & \to \mathbb{F}^{|\mathbb{F}|} \\ f & \mapsto (f(x) : x \in \mathbb{F}) \end{array}$.

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- It has relative distance $\delta_{\min}(\mathscr{D}) = 1 - \frac{\ell - 1}{\mathbb{F}}$.

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- It has relative distance $\delta_{\min}(\mathscr{D}) = 1 - \frac{\ell-1}{\mathbb{F}}$.
- Let $(f_i)_{i\in[\ell]}$ be a basis of $\mathbb{F}[x]_{<\ell}$.

Then $G_{\mathscr{D}} \colon \begin{array}{ll} \mathbb{F} & \to \mathbb{F}^{\ell} \\ x & \mapsto (f_i(x))_{i\in[\ell]} \end{array}$ is $\frac{\ell-1}{\mathbb{F}}$-biased.

Generators from MDS codes preserve distance

Key Lemma

Let $\mathcal{L} \subseteq \{\mathbb{F}^s \to \mathbb{F}\}$ be a $\mathbb{F}$-linear space and let $\{f_1, \ldots, f_\ell\}$ be a basis of $\mathcal{L}$.

Assume that $\mathscr{D} = \mathrm{ev}(\mathcal{L})$ is **MDS**, meaning $\delta_{\min}(\mathscr{D}) = 1 - \frac{\ell - 1}{N}$. $N := |\mathbb{F}^s|$

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1. ε-biased for $\varepsilon = \frac{\ell-1}{N}$,
2. (Λ, σ, τ)-distance-preserving.

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Proof of Key Lemma — Unique-decoding regime

Unique-decoding regime: $\delta < \frac{\delta_{\min}}{\ell+1}$

Assume $\exists A \subseteq \mathbb{F}^s, |A| > \tau \cdot |\mathbb{F}^s|$ s.t.
$\forall \boldsymbol{x} \in A, \Delta(G_{\mathscr{D}}(\boldsymbol{x}) \cdot \boldsymbol{U}, \mathscr{C}) < \delta$.

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- For each $\boldsymbol{x} \in A$, consider $c_{\boldsymbol{x}} \in \mathscr{C}$ that is δ-close to $G_{\mathscr{D}}(\boldsymbol{x}) \cdot \boldsymbol{U}$.

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Step 1. Find $\mathbf{C} \in \mathscr{C}^\ell$ s.t. $\forall \boldsymbol{x} \in A, \Delta(G_{\mathscr{D}}(\boldsymbol{x}) \cdot \boldsymbol{U}, G_{\mathscr{D}}(\boldsymbol{x}) \cdot \mathbf{C}) < \delta$.

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Proof of Key Lemma — Unique-decoding regime

Unique-decoding regime: $\delta < \frac{\delta_{\min}}{\ell+1}$

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- Since $\mathscr{D}$ is MDS*, compute $\mathbf{C} \in \mathscr{C}^\ell$ s.t. $\forall i \in [\ell], c_{\boldsymbol{s}_i} = G_{\mathscr{D}}(\boldsymbol{s}_i) \cdot \mathbf{C}$.

* $[N, \ell]$-code $\mathscr{D}$ is MDS iff for any $S \subseteq \mathbb{F}^s$, $|S| = \ell$, $\{G_{\mathscr{D}}(\boldsymbol{s}) : \boldsymbol{s} \in S\}$ is linearly independent.

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Proof of Key Lemma — ~~Unique-decoding regime~~ List-decoding regime

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New Step 1. Find a large subset $B \subseteq A$ and $\mathbf{C} \in \mathscr{C}^\ell$ such that
$\forall \boldsymbol{x} \in B, \Delta(G_{\mathscr{D}}(\boldsymbol{x}) \cdot \boldsymbol{U}, G_{\mathscr{D}}(\boldsymbol{x}) \cdot \mathbf{C}) < \delta$.

More challenging because codewords are very noisy.

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Proof overview

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- Consider $G_{\mathscr{D}}\colon \mathbb{F} \to \mathbb{F}^{d+1}$ where $\mathscr{D}$ is a RS code of dimension $d+1$.
- $G_{\mathscr{D}}$ preserves distance $\implies$ ditto for $G\colon \mathbb{F} \to \mathbb{F}^{\ell}$

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Any **multivariate** degree-d generator $G\colon \mathbb{F}^s \to \mathbb{F}^\ell$ is distance-preserving.

◇ By induction on the number of variables s.

Conclusion

Summary

Any polynomial generator is distance-preserving.

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- Larger proximity range Λ? Smaller error probability τ?
  - τ is sharp in some settings, e.g. $G(x) = (x^i)_{0 \le i < \ell}$ when $\delta < \delta_{\min}/2$.

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Any polynomial generator is distance-preserving.

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  - τ is sharp in some settings, e.g. $G(x) = (x^i)_{0 \le i < \ell}$ when $\delta < \delta_{\min}/2$.
- New distance-preserving generators? (Yes [AGHP92])
- Are all biased generators also distance-preserving generators?

Thanks!